B.Sc. Part I (Hons.) Trigonometry (contd.)

1st Paper

1. Sum the series

$$1 - \frac{1}{2}\cos\alpha + \frac{1.3}{2.4}\cos 2\alpha - \frac{1.3.5}{2.4.6}\cos 3\alpha + \cdots \text{ to } \infty$$

Soln

Let $C = 1 - \frac{1}{2}\cos\alpha + \frac{1.3}{2.4}\cos 2\alpha - \frac{1.3.5}{2.4.6}\cos 3\alpha + \cdots \text{ to } \infty.$

Let $S = -\frac{1}{2}\sin\alpha + \frac{1.3.}{2.4}\sin 2\alpha - \frac{1.3.5}{2.4.6}\sin 3\alpha + \cdots \text{ to } \infty$

$$\Rightarrow C + iS = 1 - \frac{1}{2}(\cos\alpha + i\sin\alpha) + \frac{1.3}{2.4}(\cos 2\alpha + i\sin 2\alpha) - \frac{1.3.5}{2.4.6}(\cos 3\alpha + i\sin 3\alpha) + \cdots \text{ to } \infty$$

$$\Rightarrow C + iS = 1 - \frac{1}{2}e^{i\alpha} + \frac{1.3}{2.4}e^{2i\alpha} - \frac{1.3.5}{2.4.6}e^{3i\alpha} + \cdots \text{ to } \infty$$

Put $e^{i\alpha} = x$

$$\Rightarrow C + iS = 1 - \frac{x}{2} + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \cdots \text{ to } \infty$$

$$= (1+x)^{-1/2}$$

$$= \left[1 + e^{i\alpha}\right]^{-1/2}$$

$$\Rightarrow C+iS = \frac{1}{(1+e^{i\alpha})^{1/2}} = \frac{1}{\sqrt{1+\cos\alpha+i\sin\alpha}}$$

$$\Rightarrow C+iS = \frac{1}{\sqrt{2\cos^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$\Rightarrow C+iS = \frac{1}{\sqrt{2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)}}$$

$$\Rightarrow C+iS = \frac{1}{\sqrt{2\cos\frac{\alpha}{2}}}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)^{-1/2}$$

$$\Rightarrow C+iS = \frac{\sqrt{2\sec\frac{\alpha}{2}}}{2}\left(\cos\frac{\alpha}{4}-i\sin\frac{\alpha}{4}\right)$$

Equating real parts, we get

$$\therefore C = \frac{\sqrt{2\sec\frac{\alpha}{2}}}{2}\cdot\cos\frac{\alpha}{4}$$

Q. Find the sum of the following series

$$n \sin\alpha + \frac{n(n+1)}{1\cdot 2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin 3\alpha + \cdots \text{ to } \infty.$$

Soln

Let $S = n\sin\alpha + \frac{n(n+1)}{1\cdot 2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin 3\alpha + \cdots$ to ∞

Let $C = 1 + n\cos\alpha + \frac{n(n+1)}{1\cdot 2}\cos 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cos 3\alpha + \cdots$ to ∞.

$$\therefore C + iS = 1 + n(\cos\alpha + i\sin\alpha) + \frac{n(n+1)}{\lfloor 2}(\cos 2\alpha + i\sin 2\alpha) + \frac{n(n+1)(n+2)}{\lfloor 3}(\cos 3\alpha + i\sin 3\alpha) + \cdots \text{ to } \infty$$

$$\Rightarrow C + iS = 1 + n e^{i\alpha} + \frac{n(n+1)}{\lfloor 2} e^{2i\alpha} + \frac{n(n+1)(n+2)}{\lfloor 3} e^{3i\alpha} + \cdots \text{ to } \infty$$

Put $e^{i\alpha} = x$

$$\Rightarrow C + iS = 1 + nx + \frac{n(n+1)}{\lfloor 2} x^2 + \frac{n(n+1)(n+2)}{\lfloor 3} x^3 + \cdots \text{ to } \infty$$

$$\Rightarrow C+iS = (1-x)^{-n}$$

$$= \left(1-e^{i\alpha}\right)^{-n}$$

$$= \left(1-\cos\alpha - i\sin\alpha\right)^{-n}$$

$$= \left(2\sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)^{-n}$$

$$\Rightarrow C+iS = \left[2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)\right]^{-n}$$

$$= \left[2\sin\frac{\alpha}{2}\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}\right]^{-n}$$

$$= 2^{-n}\left(\sin\frac{\alpha}{2}\right)^{-n}\left[\cos\left(\frac{\pi-\alpha}{2}\right) - i\sin\left(\frac{\pi-\alpha}{2}\right)\right]^{-n}$$

$$\Rightarrow C+iS = 2^{-n}\left(\sin\frac{\alpha}{2}\right)^{-n}\left[\cos\frac{n(\pi-\alpha)}{2} + i\sin\frac{n(\pi-\alpha)}{2}\right]$$

Equating imaginary parts, we get

$$\therefore S = 2^{-n}\left(\sin\frac{\alpha}{2}\right)^{-n}\sin\left(\frac{n(\pi-\alpha)}{2}\right).$$